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A probabilistic growth model for partition polygons and related structures

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Abstract

A two-parameter, probabilistic growth model for partition polygon clusters is introduced and exact results obtained relating to the area moments and the area probability distribution. In particular, the scaling behaviour in the presence of asymmetry between growth along the two principal axes is discussed. Variants of the model are also examined, including the extension to rooted stack polygons. An interesting application relates to characterizing the asymptotic behaviour of the cumulative customer waiting time distribution in a particular discrete-time queue.

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1. Introduction

Partition polygons are a class of self-avoiding polygons defined on a square lattice [1]. Despite their apparent simplicity, they have a rich structure which is of fundamental importance in the theory of number partitions (so-called Ferrers diagrams) [2], and also of interest in the context of lattice models of crystal surfaces and vesicle behaviour [3, 4]. Here, a two-parameter, probabilistic growth model for compact clusters that take the shape of partition polygons (and related structures) is proposed and solved. The motivation for the work, following on from [5], comes from trying to identify classes of exactly solvable cluster growth models, in the spirit that these often shed valuable light on more complex, as yet unsolved, models. It must be conceded that the resulting growth models are somewhat artificial in their natural context. However, they do provide a new and interesting connection to queueing theory, as explained at the end of the paper.

To define the basic problem, consider a single square which will act as a root site and which is assumed to be present with probability one (figure 1). Growth proceeds by adding squares horizontally and vertically, such that every new row contains a square on top of every square in the row below plus possible overhangs on the right (of arbitrary length). Regarding

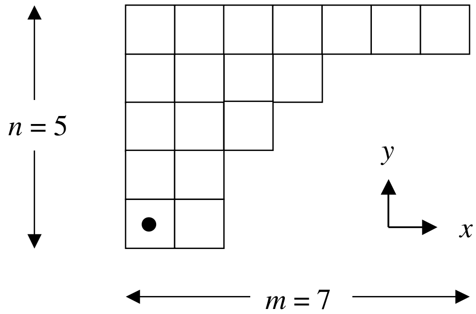

Figure 1. A typical partition polygon cluster (Ferrers diagram) of height $n = 5$ and width $m = 7$, grown from a single root square (containing the solid circle). The activity of the polygon is $x^{14}\, y^{10}\, z^{18}$, and the probabilistic weight is $p_x^6\, p_y^4\, q_x^5\, q_y$.

the growth sequence, growth along a given row proceeds until such time as it terminates; then the next row is added and so on. The growth rules are that each overhanging square within a row (except for the root) occurs with probability p_x, and each new row (apart from the first) occurs with probability p_y. The resulting structure, once growth terminates, is a partition polygon (figure 1) with probabilistic weight $p_x^{M-1} p_y^{N-1} q_x^N q_y$, where $q_x \equiv 1 - p_x$, $q_y \equiv 1 - p_y$ and N, M are the number of rows and columns respectively. The factors of q_x^N and q_y relate to terminating the growth process in the horizontal and vertical directions respectively. Questions of interest relate to the cluster size moments and to the probability distribution for generating clusters of a given size.

It transpires that in this model the cluster moments diverge only on the critical lines $p_x = 1$ and $p_y = 1$. Thus the phase diagram is simpler in structure than for models based on, for example, compact directed percolation, where the critical line is given by $p_x + p_y = 1$ [5]. However, it remains a particular challenge to understand the scaling behaviour near these lines in the presence of asymmetry (competition) between growth in the horizontal and vertical directions. Such features are not often explored but in the present model can be handled exactly in many respects.

2. Generating functions

To begin the analysis, a slightly unconventional approach to deriving the key generating functions of interest is adopted. The benefit is felt later when the formulae obtained are used to derive new results for related models such as the rooted stack polygon model; see, e.g., (33). Let $g_{m,n}(x, y, z)$ be the generating function for enumerating partition polygons that are restricted to having *fixed* width m and height n, where x and y are the horizontal and vertical perimeter activities and z is the area activity (figure 1). Clearly $g_{m,n}(x, y, z) = x^{2m} y^{2n} f_{m,n}(z)$. The lower boundary of any such polygon can be generated by taking a directed walk of $m + n$ steps from lower left to top right, where the first step is horizontal, the last step is vertical, and the order of the remaining $m - 1$ horizontal steps and $n - 1$ vertical steps is unique to the polygon in question. This picture provides a useful combinatorial decomposition of partition polygons based on the two possible choices for the second step of the walk (see figure 2). As a result, $f_{m,n}(z)$ obeys the following recursion relation,

$$f_{m,n}(z) = z f_{m,n-1}(z) + z^n f_{m-1,n}(z) \tag{1}$$

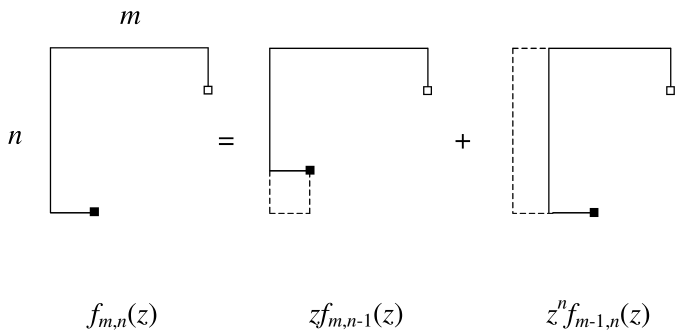

Figure 2. A decomposition for $f_{m,n}(z)$, based upon the two possible choices (vertical or horizontal) for the second perimeter step at the lower vertex (filled square) in the left-hand diagram. This defines a recursive structure for the generating function.

with $f_{m,n}(z) = f_{n,m}(z)$ and $f_{1,n}(z) = z^n$. Setting $m = 1$ in (1) shows that one must take $f_{0,n}(z) \equiv 0$ for consistency. In general, $f_{m,n}(z)$ is a finite polynomial in z whose coefficients count the number of ways to partition an integer s into exactly n (or m) parts of which the largest is m (or n). For $z = 1$ the solution of (1) is given by

$$f_{m,n}(1) = \frac{(m+n-2)!}{(m-1)!(n-1)!} \tag{2}$$

which has a natural interpretation in terms of the directed walk argument above. Without having to find $f_{m,n}(z)$ for $z \neq 1$ explicitly, one can sum (1) over m to derive

$$G_n(z) = zG_{n-1}(z) + z^n G_n(z) \qquad G_n(z) \equiv \sum_{m=1}^{\infty} f_{m,n}(z) \qquad G_0(z) \equiv 1. \tag{3}$$

The solution of this first-order recursion is simply

$$G_n(z) = z^n \prod_{i=1}^{n} \frac{1}{(1-z^i)} \qquad n \geqslant 1 \tag{4}$$

whereupon it follows that

$$G(z) \equiv \sum_{n=1}^{\infty} G_n(z) = \prod_{i=1}^{\infty} \frac{1}{(1-z^i)} - 1. \tag{5}$$

This well-known expression counts the total number of partitions of a given integer [1, 2]. The full partition polygon area–perimeter generating function comes from re-writing (1) in terms of $g_{m,n}(x, y, z)$ and summing over m then n,

$$G(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} g_{m,n}(x, y, z) = x^2 \sum_{n=1}^{\infty} y^{2n} z^n \prod_{i=1}^{n} \frac{1}{(1-x^2 z^i)}. \tag{6}$$

It is useful to note on grounds of symmetry that this expression is invariant under the interchange $x \leftrightarrow y$. Setting $z = 1$ in (6) gives the perimeter generating function

$$G(x, y, 1) = \frac{x^2 y^2}{1-(x^2+y^2)}. \tag{7}$$

It is evident that $G(x, y, z)$ is singular on a locus of critical points defined by $x^2 + y^2 = 1$ and $z = 1$. More specifically, $G(x, y, z)$ is convergent if $z < 1$ for all $x, y \leqslant 1$, and also convergent if $z = 1$ provided that $x^2 + y^2 < 1$ [4].

3. Moments and the asymptotic behaviour

To incorporate probability, the key observation is that the weight of a given partition polygon is, within the prescribed model, determined by its perimeter activity. Setting $z = 1$, $x = \sqrt{p_x}$ and $y = \sqrt{p_y q_x}$ in the polygon activity, and multiplying by a global factor of $q_y(p_x p_y)^{-1}$, generates the correct probabilistic weighting of every polygon, as may easily be verified (see figure 1). It follows using (6) that the area probability generating function is given by

$$\hat{G}(p_x, p_y, z) = q_y \sum_{n=1}^{\infty} p_y^{n-1} q_x^n z^n \prod_{i=1}^{n} \frac{1}{(1 - p_x z^i)} \equiv \sum_s P_s(p_x, p_y) z^s \tag{8}$$

where s is the polygon area (size) and $P_s(p_x, p_y)$ is the corresponding probability. Note that $\hat{G}(p_x, p_y, 1) = 1$, as required. The area moments are given in turn by

$$S_k \equiv \langle s^k \rangle = \left(z \frac{\partial}{\partial z} \right)^k \hat{G}(p_x, p_y, z) \bigg|_{z=1} \tag{9}$$

with first moment (mean area)

$$S_1 = \frac{p_x}{(1 - p_x)(1 - p_y)^2} + \frac{1}{1 - p_y}. \tag{10}$$

This expression is *exact* for all values of $p_x, p_y < 1$. It diverges when either $p_y \to 1$ or $p_x \to 1$, which accords with one's expectations, although the exponents differ in each case. The limits $p_x, p_y \to 0$ are self-explanatory. Higher order moments can also be derived exactly, although evaluating the necessary derivatives of (8) becomes progressively harder. One can, however, make statements about the asymptotic or scaling behaviour of the area moments and the area probability distribution, and these are of interest because of the two-parameter nature of the problem. The two principal limiting cases will be considered separately. It is useful to note for later that, based on a direct calculation,

$$S_2(p_x, p_y \to 1) \sim \frac{6p_x^2}{q_x^2 q_y^4} \qquad S_2(p_x \to 1, p_y) \sim \frac{2 + 4p_y}{q_x^2 q_y^4}. \tag{11}$$

3.1. The limit $p_y \to 1$

In this section the limit $p_y \to 1$ with $p_x \neq 0, 1$ fixed is considered. Rather than directly manipulating the exact solution (8), an alternative approach ultimately proves to be more useful in terms of analysing the critical behaviour. First, one can exploit the symmetry $G(x, y, z) = G(y, x, z)$ to rewrite (6) and hence (8) as

$$\hat{G}(p_x, p_y, z) = q_x q_y \sum_{n=1}^{\infty} p_x^{n-1} z^n \prod_{i=1}^{n} \frac{1}{(1 - p_y q_x z^i)}. \tag{12}$$

Define $\widetilde{G}(p_x, p_y, z) \equiv (q_x q_y)^{-1} \hat{G}(p_x, p_y, z)$. Inspection of (12) reveals that this modified generating function obeys a q-linear functional equation [6],

$$\widetilde{G}(p_x, p_y, z) = \frac{z}{1 - p_y q_x z} + \frac{p_x z}{1 - p_y q_x z} \widetilde{G}(p_x, p_y z, z) \tag{13}$$

which can be rewritten as

$$\widetilde{G}(p_x, p_y, z) = z + p_y q_x z \widetilde{G}(p_x, p_y, z) + p_x z \widetilde{G}(p_x, p_y z, z). \tag{14}$$

From the discussion in section 2 it follows that the generating function is singular in the limit $z \to 1^-$ and $p_y \to 1$. Scaling arguments [4, 6, 7] suggest that in this limit the generating function has the following form,

$$\widetilde{G}(p_x, p_y, z) \sim \frac{1}{\varepsilon^\theta} F_y\left(\frac{1-p_y}{\varepsilon^\varphi}, p_x\right) \tag{15}$$

where $\varepsilon \equiv 1-z$ and p_x is a fixed parameter. To evaluate the scaling function $F_y(\tau, p_x)$, where $\tau = (1-p_y)\varepsilon^{-\varphi}$, one expands (14) around the critical point using the method of dominant balance [6, 7]. This entails letting $p_y \to 1$ and, simultaneously, $\varepsilon \to 0^+$ such that τ remains fixed, i.e. the evolution of the variable p_y is controlled by the defining relation $p_y \equiv 1 - \tau\varepsilon^\varphi$. The result, after inserting (15) into (14), expanding and retaining the leading order (dominant) terms in powers of ε, is that

$$0 = 1 - q_x\tau\varepsilon^{\varphi-\theta}F_y(\tau, p_x) + p_x\varepsilon^{1-\varphi-\theta}\frac{\mathrm{d}F_y(\tau, p_x)}{\mathrm{d}\tau} + \cdots. \tag{16}$$

For there to be a non-trivial scaling function the exponents θ and φ must take the values $\theta = \varphi = 1/2$, whereupon $F_y(\tau, p_x)$ obeys

$$p_x\frac{\mathrm{d}F_y(\tau, p_x)}{\mathrm{d}\tau} = q_x\tau F_y(\tau, p_x) - 1. \tag{17}$$

This has the solution

$$F_y(\tau, p_x) = \sqrt{\frac{\pi}{2p_xq_x}}\operatorname{erfc}\left(\left(\frac{q_x}{2p_x}\right)^{1/2}\tau\right)\exp\left(\frac{q_x}{2p_x}\tau^2\right) \tag{18}$$

and this expression ultimately determines the asymptotic scaling behaviour of the probability generating function (12) in the limit $z \to 1^-$ and $p_y \to 1$.

For $\tau \to \infty$, i.e. $z \to 1^-$ with $p_y \sim 1$ fixed, an asymptotic expansion of (18) leads to the following useful result,

$$\hat{G}(p_x, p_y, z) \sim \sum_{n\geqslant 0}(-1)^n\frac{\Gamma\left(n+\frac{1}{2}\right)2^n}{\sqrt{\pi}}\frac{p_x^n}{q_x^nq_y^{2n}}(1-z)^n. \tag{19}$$

From this one can work out the asymptotic behaviour of the moments by noting in relation to (9)

$$S_k(p_x, p_y \to 1) \sim (-1)^k\left(\frac{\partial}{\partial\varepsilon}\right)^k\hat{G}(p_x, p_y, 1-\varepsilon)\bigg|_{\varepsilon=0}. \tag{20}$$

The result is

$$S_k(p_x, p_y \to 1) \sim \frac{(2k)!}{2^k}\left(\frac{p_x}{q_x}\right)^k\frac{1}{(1-p_y)^{2k}}. \tag{21}$$

Note that (10) and (11) are consistent with this expression. Concerning the probability distribution itself, using (15) and (18) one has for $p_y \to 1$ and $z \to 1^-$,

$$\sum_s P_s(p_x, p_y \to 1)z^s \sim \left(\frac{\pi\alpha}{\varepsilon}\right)^{1/2}\operatorname{erfc}\left(\left(\frac{\alpha}{\varepsilon}\right)^{1/2}\right)\exp\left(\frac{\alpha}{\varepsilon}\right) \tag{22}$$

where $\alpha = q_xq_y^2\big/2p_x$. Since $z^s \sim \exp(-\varepsilon s)$, the transform can be inverted (care is needed to ensure the procedure is rigorous) to give an asymptotic form valid for $p_y \to 1$ and $s \to \infty$,

$$P_s(p_x, p_y \to 1) \sim \left(\frac{q_x}{2p_x}\right)^{1/2}(1-p_y)s^{-1/2}\exp\left(-\left(\frac{2q_x}{p_x}\right)^{1/2}(1-p_y)s^{1/2}\right). \tag{23}$$

This is an interesting result. Contrary, perhaps, to one's expectation, the tail of the distribution is heavy (Weibull). A consistency check comes from multiplying (23) by s^k and integrating to re-derive the asymptotic behaviour of the cluster moments (21). The scaling form (23) specifies the asymptotic behaviour of the model as $p_y \to 1$ for $p_x \neq 0, 1$ fixed.

3.2. The limit $p_x \to 1$

The limit $p_x \to 1$, with $p_y \neq 0, 1$ fixed, is slightly less amenable to analytic treatment than the limit $p_y \to 1$. However, the method of dominant balance may still be used to analyse (14). The appropriate scaling form this time is

$$\widetilde{G}(p_x, p_y, z) \sim \frac{1}{\varepsilon} F_x \left(\frac{1-p_x}{\varepsilon}, p_y \right) \tag{24}$$

with the scaling function, $F_x(\tau, p_y)$, where $\tau = (1-p_x)\varepsilon^{-1}$, obeying the following differential equation,

$$p_y \frac{\partial F_x(\tau, p_y)}{\partial p_y} = 1 - (1 + \tau q_y) F_x(\tau, p_y). \tag{25}$$

The solution of (25) may be written in the form

$$F_x(\tau, p_y) = \mathrm{e}^{\tau p_y} \int_0^1 t^\tau \, \mathrm{e}^{-\tau p_y t} \, \mathrm{d}t = \mathrm{e}^{\tau p_y} (\tau p_y)^{-\tau-1} \gamma(\tau + 1, \tau p_y) \tag{26}$$

where $\gamma(a, b)$ is the incomplete gamma function. Despite knowing (26), it has not proved possible to analytically invert the generating function (24) to obtain the probability distribution, and this remains an open problem. However, there exists an asymptotic expansion of (26) in powers of ε of the following form,

$$F_x \sim \sum_{n \geqslant 0} (-1)^n \frac{C_n(p_y)}{q_x^{n+1} q_y^{2n+1}} \varepsilon^{n+1} \qquad C_0 \equiv 1 \tag{27}$$

whose coefficients determine the asymptotic behaviour of the moments,

$$S_k(p_x \to 1, p_y) \sim \frac{k!}{(1-p_x)^k} \frac{C_k(p_y)}{q_y^{2k}}. \tag{28}$$

Substituting (27) into (25) it follows in straightforward fashion that

$$C_k = p_y q_y \frac{\partial C_{k-1}}{\partial p_y} + [1 + 2p_y(k-1)] C_{k-1} \qquad k \geqslant 2 \tag{29}$$

with $C_1 = 1$. Equation (29) can be solved recursively, e.g. for $k = 2$ one has $C_2 = 1 + 2p_y$, whereupon (28) agrees with (11). Moreover, as $p_y \to 1$ one finds that $C_k \to (2k)!/k!2^k$, so that (28) is fully consistent with (21) in the limit $p_x \to 1$. In general, the required solution for C_k is a polynomial in p_y whose highest order term is $k!p_y^{k-1}$. The structure of C_k hints at a more complex form for the unknown probability distribution than the comparatively simple scaling result (23).

4. Variants of the basic model

Having studied the basic model one can look at related problems. Suppose that the polygons are conditioned so that they always have a fixed height, N, which can form the basis of a growth model in a restricted geometry. The relevant generating function is given by using the results in section 2,

$$G_N(x, y, z) = y^{2N} \sum_{m=1}^{\infty} x^{2m} f_{m,N}(z) = x^2 y^{2N} z^N \prod_{i=1}^{N} \frac{1}{(1 - x^2 z^i)}. \tag{30}$$

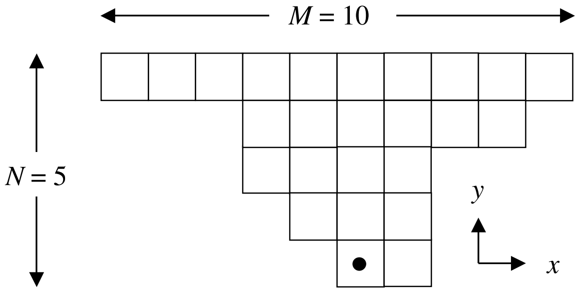

Figure 3. A typical rooted stack polygon cluster with $N = 5$ rows and $M = 10$ columns. The root column contains the solid circle. The activity of this particular stack polygon is $x^{20}\, y^{10}\, z^{25}$, and the probabilistic weight is $p_x^9\, p_y^4\, q_x^{10}\, q_y$.

The corresponding area probability generating function (taking into account the conditional nature of the problem) is

$$\hat{G}_N(p_x, z) = q_x^N z^N \prod_{i=1}^{N} \frac{1}{(1 - p_x z^i)}. \tag{31}$$

This expression no longer depends upon p_y since N is fixed. For the mean area,

$$S_1 \equiv \left(z \frac{\partial \hat{G}_N}{\partial z} \right)\bigg|_{z=1} = N + \frac{p_x}{1 - p_x} \frac{N(N+1)}{2}. \tag{32}$$

Setting $N = 1$ in this result is equivalent to setting $p_y = 0$ in (10). When $p_x = 0$ only the root column remains, whose height, N, is fixed by construction.

The initial growth model can also be modified so as to result in the formation of (rooted) stack polygons [1, 4], where the overhangs can occur on *either side* of the root column (figure 3). It is evident that these stack polygons can be represented as the concatenation (overlap) of two partition polygons, the overlap occurring for the root column which, necessarily, has the same height for each partition polygon (figure 3). Of course, a correction has to be made to prevent over-counting of area and boundary elements. The generating function for rooted stack polygons can thus be written (using results from section 2) as

$$\begin{aligned} H(x, y, z) &= \sum_{n=1}^{\infty} \sum_{r,r'=1}^{\infty} \frac{x^{2r+2r'} y^{4n} f_{r,n}(z) f_{r',n}(z)}{x^2 y^{2n} z^n} \\ &= x^2 \sum_{n=1}^{\infty} y^{2n} z^n \prod_{i=1}^{n} \frac{1}{(1 - x^2 z^i)^2} \end{aligned} \tag{33}$$

where the $x^2 y^{2n} z^n$ factor in the denominator takes care of over-counting arising from the numerator. This expression is different from the generating function for unrooted stack polygons [1, 4]. To introduce probability one now has to set $x = \sqrt{p_x}$ and $y = q_x \sqrt{p_y}$ and multiply by a global factor of $q_y (p_x p_y)^{-1}$. Note the modification to y (compared to the partition polygon case) since growth terminates on both sides. From (33), the area probability generating function is given by

$$\hat{H}(p_x, p_y, z) = q_y \sum_{n=1}^{\infty} p_y^{n-1} q_x^{2n} z^n \prod_{i=1}^{n} \frac{1}{(1 - p_x z^i)^2}. \tag{34}$$

Thus the mean area is

$$S_1 \equiv \left(z \frac{\partial \hat{H}}{\partial z} \right) \bigg|_{z=1} = \frac{2p_x}{(1-p_x)(1-p_y)^2} + \frac{1}{1-p_y}. \tag{35}$$

This result has a natural interpretation. If one represents the stack polygon area $s = s_1 + s_2 - n$, where s_1 and s_2 are the areas of the two overlapping partition polygons and n is the area of the overlap (root) column, then (35) simply states that the expected area is twice the expected area for a partition polygon (10) minus the expected area of the root column, the latter being $(1-p_y)^{-1}$. For higher order moments, however, there is no such simple relationship, due to coupling through the root column.

One can repeat the analysis of section 3 for stack polygon clusters to work out the asymptotic behaviour. Of course, the critical scaling behaviour is qualitatively similar, evidenced, for example, in the following two results (cf (23) and (21)),

$$P_s(p_x, p_y \to 1) \sim \left(\frac{q_x}{4p_x}\right)^{1/2} (1-p_y)s^{-1/2} \exp\left(- \left(\frac{q_x}{p_x}\right)^{1/2} (1-p_y)s^{1/2} \right) \tag{36}$$

$$S_k(p_x, p_y \to 1) \sim (2k)! \left(\frac{p_x}{q_x}\right)^k \frac{1}{(1-p_y)^{2k}}. \tag{37}$$

One can also examine the variant of the stack problem wherein the clusters always grow to a fixed height N. In such circumstances, the probability generating function is given by

$$\hat{H}_N(p_x, z) = \frac{q_x^{2N}}{p_x^2 z^N} \left[\sum_{m=1}^{\infty} p_x^m f_{m,N}(z) \right]^2 = z^N q_x^{2N} \prod_{i=1}^{N} \frac{1}{(1-p_x z^i)^2} \tag{38}$$

which, again, is independent of p_y. The calculation of the mean area proceeds as before with the result that

$$S_1 = \left(z \frac{\partial \hat{H}_N}{\partial z} \right) \bigg|_{z=1} = N + \frac{p_x}{1-p_x} N(N+1). \tag{39}$$

Once more this result can be interpreted as being twice the expected area of the corresponding partition polygon (32) minus the area of the root column, N. A completely different derivation of (39) was given in [5] by considering a special limit of a cluster growth model that is related to compact directed percolation.

5. An application in queueing theory

As a final (and rather interesting) topic, the following problem in queueing theory maps onto the results described above for partition polygons (see figure 4). Consider a discrete-time queue with unit time-step which operates in batch mode through a single server. Arrivals and departures occur at the time slot boundaries. At time $t = 0$, a number of customers $N > 0$ arrive into the queue, where N is distributed geometrically; $P_N = \lambda^{N-1}\overline{\lambda}$, with $\overline{\lambda} = 1 - \lambda$. These customers are then served as follows: if at time $t > 0$ the number of customers left in the queue is N_t, the number M_t departing the queue is geometrically distributed as $P_{M_t} = \mu^{M_t}\overline{\mu}$ for $0 \leqslant M_t < N_t$, with $P_{N_t} = \mu^{N_t}$, and $\overline{\mu} \equiv 1 - \mu$. It is assumed in this problem that no other customers enter the queue after the initial batch, and that the queue empties in finite time (i.e. $\mu \neq 0$). As an illustration of such a process, imagine a cohort of passengers leaving an aircraft and clearing immigration in variable size family groupings before the next

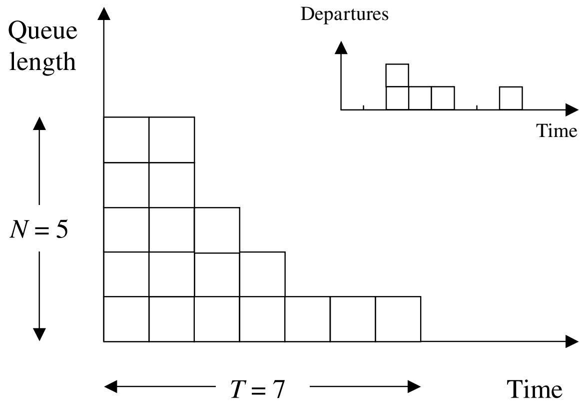

Figure 4. A partition polygon representation of queueing dynamics, with an initial random batch of $N = 5$ customers being processed over a busy period of $T = 7$. The activity of the polygon is $x^{14}\, y^{10}\, z^{18}$, and its probabilistic weight is $\lambda^4\overline{\lambda}\mu^5\overline{\mu}^6$. The cumulative customer delay is $S = 18$.

aircraft cohort arrives. Figure 4 (a simple inversion of figure 1) shows that the cumulative (i.e. total) customer delay, S, experienced in this model is just the area of a partition polygon. At a fundamental level, this is equivalent to analysing the polygon structure on a column by column basis rather than a row by row basis. As a consequence, results such as (10), (21) and (23) apply immediately with the replacements $p_x \to \overline{\mu}$ and $p_y \to \lambda$. So, for example, for $\lambda \to 1$ and $x \to \infty$,

$$\Pr(S > x) \sim \exp\left(-\left(\frac{2\mu}{\overline{\mu}}\right)^{1/2}(1-\lambda)x^{1/2}\right). \tag{40}$$

It is significant that the cumulative delay distribution in this model has a heavy tail, something which is also evident in the study of other area-like variables in queueing theory [8, 9]. In a different scenario, where the number of customers, N, arriving at $t = 0$ is fixed, one may adapt (31) so that quantities of interest can be obtained from the defining generating function

$$\hat{G}_N(\mu, z) \equiv \sum_S P_S(\mu) z^S = \prod_{i=1}^{N} \frac{\mu z}{(1 - \overline{\mu} z^i)}. \tag{41}$$

With suitable modification, one can also obtain information about the time taken by the server to service all the customers (the so-called busy period, T; see figure 4), and to cover the case where at most one customer can leave the queue per time slot. These and other issues will be addressed more fully elsewhere.

6. Conclusions

In summary, a two-parameter, probabilistic growth model for partition polygons has been defined and solved exactly for the mean area and for certain aspects of the asymptotic behaviour of the area moments and the scaling behaviour of the area probability distribution. The model allows an investigation to be made of scaling behaviour in the presence of asymmetry between growth along the two principal axes. The method of solution also allows results to be obtained for a corresponding model wherein growth in the vertical direction is restricted to a fixed height. Extensions to the case of rooted stack polygons are similarly straightforward. Finally, a simple mapping enables the results to be applied to an interesting problem in discrete-time queueing theory with batch arrivals and departures. This suggests that there may well be other problems in a variety of fields to which the present results might find immediate application.

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